10 oct - LEC 4

GOODNESS OF FIT (PT. 2)

We are considering the simple linear model $Y_i = \beta_1 + \beta_2 x_i + \varepsilon_i$, $\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$
and the system of hypotheses

$$\begin{cases} H_0: \text{the model does not help to explain the variability of } Y \\ H_1: \text{the model helps to explain the variability of } Y \end{cases}$$

which can be expressed in terms of the coefficient R^2 as

$$\begin{cases} H_0: R^2 = 0 \\ H_1: R^2 > 0 \end{cases}$$

We have seen that we can use the test statistic $(n-2) R^2 / (1-R^2)$, which, under H_0, has an $F_{1, n-2}$ distribution.

$$F = \frac{R^2}{1-R^2} \cdot (n-2) = \frac{SSR}{SSE}(n-2) =$$

$$= \left(\frac{SST}{SSE} - 1\right)(n-2) =$$

$$= \left(\frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{\sum_{i=1}^n (Y_i - \hat{Y}_i)^2} - 1 \right)(n-2) =$$

$$= \frac{\tilde{\sigma}^2 - \hat{\sigma}^2}{\hat{\sigma}^2}(n-2) \overset{H_0}{\sim} F_{1, n-2}$$

In the case of the SIMPLE linear model, we can prove this result

> Preliminary result:
>
> If $T \sim t_n$, and $V = T^2$ then $V \sim F_{1,n}$

PROOF FOR THE CASE of SIMPLE LM : distribution of F

Let's start from $\frac{SSR}{SSE} = \frac{\sum_{i=1}^n \varepsilon_i^{*2}}{\sum_{i=1}^n \varepsilon_i^2}$ with $\varepsilon_i^* = Y_i - \bar{Y}$, $\varepsilon_i = Y_i - \hat{Y}_i$

Now, notice that we can write

$$\sum_{i=1}^n \varepsilon_i^2 = \sum_{i=1}^n (Y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i)^2 = \sum_{i=1}^n (Y_i - \bar{Y} + \hat{\beta}_2 \bar{x} - \hat{\beta}_2 x_i)^2 =$$

$$= \sum_{i=1}^n [(Y_i - \bar{Y}) - \hat{\beta}_2 (x_i - \bar{x})]^2 =$$

$$= \sum_{i=1}^n (Y_i - \bar{Y})^2 + \hat{\beta}_2^2 \sum_{i=1}^n (x_i - \bar{x})^2 - 2\hat{\beta}_2 \sum_{i=1}^n (Y_i - \bar{Y})(x_i - \bar{x})$$

$$-\hat{\beta}_2 \overbrace{\frac{\sum_{i=1}^n (Y_i - \bar{Y})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}}^{\hat{\beta}_2} \cdot \sum_{i=1}^n (x_i - \bar{x})^2 = -\hat{\beta}_2^2 \sum_{i=1}^n (x_i - \bar{x})^2$$

$$= \sum_{i=1}^n \underbrace{(Y_i - \bar{Y})^2}_{\varepsilon_i^{*2}} - \hat{\beta}_2^2 \sum_{i=1}^n (x_i - \bar{x})^2$$

$$\Rightarrow \sum_{i=1}^n \varepsilon_i^2 = \sum_{i=1}^n \varepsilon_i^{*2} - \hat{\beta}_2^2 \sum_{i=1}^n (x_i - \bar{x})^2$$

$$\Rightarrow \sum_{i=1}^n \varepsilon_i^{*2} = \sum_{i=1}^n \varepsilon_i^2 + \hat{\beta}_2^2 \sum_{i=1}^n (x_i - \bar{x})^2$$

Moreover, recall that

$$V(\hat{\beta}_2) = \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \quad ; \quad \hat{V}(\hat{\beta}_2) = \frac{S^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \quad ; \quad \frac{(n-2)S^2}{\sigma^2} \sim \chi^2_{n-2}$$

Going now back to the test statistic

$$\frac{R^2}{1-R^2} = \frac{\sum_{i=1}^n \varepsilon_i^{*2}}{\sum_{i=1}^n \varepsilon_i^2} - 1 = \frac{\sum_{i=1}^n \varepsilon_i^2 + \hat{\beta}_2^2 \sum_{i=1}^n (x_i - \bar{x})^2}{\sum_{i=1}^n \varepsilon_i^2} - 1 =$$

$$= \cancel{1} + \frac{\hat{\beta}_2^2 \sum_{i=1}^n (x_i - \bar{x})^2}{\sum_{i=1}^n \varepsilon_i^2} - \cancel{1} =$$

$$\sum_{i=1}^n \varepsilon_i^2 = \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 = n\hat{\sigma}^2 = (n-2)S^2$$

$$= \frac{\hat{\beta}_2^2 \sum_{i=1}^n (x - \bar{x})^2}{(n-2)S^2} =$$

$$= \frac{\hat{\beta}_2^2 \cdot \frac{1}{\sigma^2}}{\left((n-2) \frac{S^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right) \cdot \frac{1}{\sigma^2}} =$$

$$= \frac{\dfrac{\hat{\beta}_2^2}{\left(\frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right)}}{\dfrac{(n-2) S^2}{\sigma^2}} = \frac{\dfrac{\hat{\beta}_2^2}{\text{var}(\hat{\beta}_2)}}{\dfrac{(n-2)S^2}{\sigma^2} \cdot \dfrac{1}{(n-2)}} \cdot \frac{1}{(n-2)}$$

$$= \frac{\overbrace{\left(\dfrac{\hat{\beta}_2}{\text{var}(\hat{\beta}_2)} \right)^2}^{\overset{H_0}{\sim} N(0,1)^2}}{\underbrace{\dfrac{(n-2)S^2}{\sigma^2} \cdot \dfrac{1}{(n-2)}}_{\overset{H_0}{\sim} \frac{\chi^2_{n-2}}{n-2}}} \cdot \frac{1}{(n-2)} = T^2 \cdot \frac{1}{n-2}$$

where $T = \frac{\hat{\beta}_2 \sqrt{\sum_{i=1}^n (x - \bar{x})^2}}{\sqrt{S^2}} = \frac{\hat{\beta}_2}{\sqrt{\frac{S^2}{\sum_{i=1}^n (x_i - \bar{x})^2}}} = \frac{\hat{\beta}_2}{\sqrt{\hat{\text{var}}(\hat{\beta}_2)}} \overset{H_0}{\sim} t_{n-2}$

$$\Rightarrow F = \frac{R^2}{1-R^2} \cdot (n-2) = T^2 \overset{H_0}{\sim} F_{1, n-2}$$

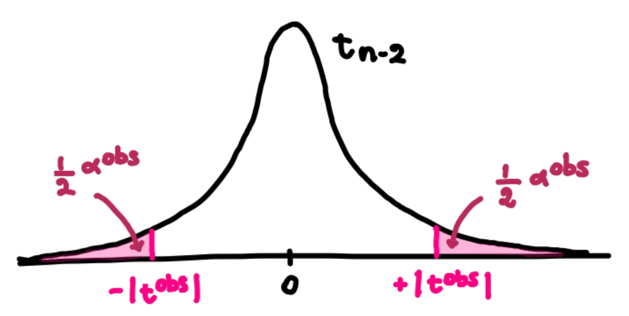
So, we have derived the distribution of the test statistic (in the case of simple lm).

Remark: connection with the p-value of the test $H_0: \beta_2 = 0$ vs $H_1: \beta_2 \neq 0$

$$\mathbb{P}_{H_0}(F \geq f^{obs}) = \mathbb{P}_{H_0}(T^2 \geq (t^{obs})^2)$$

$$= \mathbb{P}_{H_0}(|T| \geq |t^{obs}|) =$$

$$= 2\, \mathbb{P}_{H_0}(T \geq |t^{obs}|) \quad T \overset{H_0}{\sim} t_{n-2}$$



where T is exactly the test statistic we derived to test β_2