24 OCT - LEC 6

PROPERTIES OF THE ESTIMATORS

Given $\underline{Y} = (Y_1, \dots, Y_n)$ and X $(n \times p)$

we have seen that the estimators are
$$\hat{\underline{\beta}} = (X^T X)^{-1} X^T \underline{Y}$$
$$\hat{\Sigma}^2 = \frac{1}{n} \underline{E}^T \underline{E}$$
$$S^2 = \frac{1}{n-p} \underline{E}^T \underline{E}$$

we want to derive their exact distribution to perform inference.

• DISTRIBUTION OF $\hat{\underline{\beta}}$

estimator $\hat{\underline{\beta}} = (X^T X)^{-1} X^T \underline{Y}$ linear in $\underline{Y}$ (i.e. $\hat{\underline{\beta}} = A \cdot \underline{Y}$) with $\underline{Y} \sim N_n(X\underline{\beta}, \sigma^2 I_n)$

• LINEAR TRANSFORMATIONS of GAUSSIAN RANDOM VECTORS

$\underline{z} \sim N_d(\underline{\mu}, \Sigma)$, A $(k \times d)$ matrix, $\underline{b} \in \mathbb{R}^k$

$$\Rightarrow \quad T = \underbrace{A}_{(k\times d)} \underbrace{\underline{z}}_{(d\times 1)} + \underbrace{\underline{b}}_{(k\times 1)} \sim N_k(A\underline{\mu} + \underline{b}, A \Sigma A^T)$$

$$\mathbb{E}[\underline{T}] = \mathbb{E}[A\underline{z} + \underline{b}] = A\,\mathbb{E}[\underline{z}] + \underline{b} = A\underline{\mu} + \underline{b}$$
$$\mathrm{var}(\underline{T}) = \mathrm{var}(A\underline{z} + b) = \mathrm{var}(A\underline{z}) = \mathbb{E}[(A\underline{z} - \mathbb{E}[A\underline{z}])(A\underline{z} - \mathbb{E}[A\underline{z}])^T]$$
$$= \mathbb{E}[(A\underline{z} - A\underline{\mu})(A\underline{z} - A\underline{\mu})^T] = \mathbb{E}[A(\underline{z}-\underline{\mu})(\underline{z}-\underline{\mu})^T A^T] = A \Sigma A^T$$

$\Rightarrow \hat{\underline{\beta}} = A\underline{Y}$ with $A = (X^T X)^{-1} X^T$ $p \times n$ rank $= p$

$$\mathbb{E}[\hat{\underline{\beta}}] = A\,\mathbb{E}[\underline{Y}] = (X^T X)^{-1} X^T X \underline{\beta} = \underline{\beta}$$
$$\mathrm{var}(\hat{\underline{\beta}}) = A\,\mathrm{var}(\underline{Y}) A^T = A(\sigma^2 I_n) A^T = \sigma^2 A A^T = \sigma^2 \cancel{(X^T X)^{-1}} \cancel{X^T X} ((X^T X)^{-1})^T = \sigma^2 (X^T X)^{-1}$$

$$\Rightarrow \quad \hat{\underline{\beta}} \sim N_p(\underline{\beta}; \sigma^2 (X^T X)^{-1})$$

the marginal distribution of the j-th component is $\hat{\beta}_j \sim N_1(\beta_j; \sigma^2 [(X^T X)^{-1}]_{(j,j)})$ $\qquad j = 1, \dots, p$

the covariance between two components j and s $\mathrm{cov}(\hat{\beta}_j, \hat{\beta}_s) = \sigma^2 [(X^T X)^{-1}]_{(j,s)}$ $\qquad j, s = 1, \dots, p$

Notice that the variance is $\sigma^2 (X^T X)^{-1}$: once again, we need $(\underline{x}_1, \dots, \underline{x}_p)$ to be linearly independent.

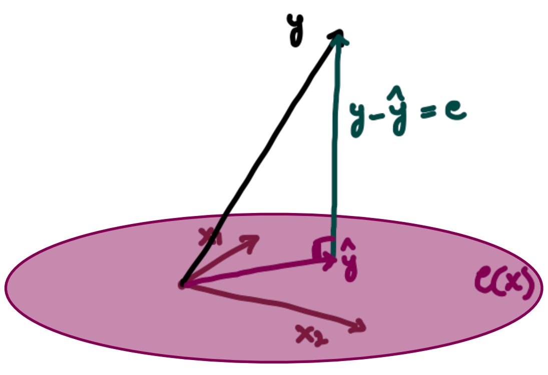
indeed $(X^T X)^{-1} = \frac{1}{\det(X^T X)} \cdot [\dots]$

If they are linearly dependent, $\det(X^T X) = 0$ and $(X^T X)$ is not invertible

Remark: if they are "almost collinear" $(\det(X^T X) \approx 0)$ the variance of the estimator explodes

$$\mathrm{var}(\hat{\underline{\beta}}) = \sigma^2 \cdot \frac{1}{\det(X^T X)} \cdot [\dots]$$
$\to +\infty$

• DISTRIBUTION OF THE RESIDUALS



$$\underline{e} = \underline{y} - \hat{\underline{y}} = (I_n - P)\underline{y}$$

projection of $\underline{y}$ onto the subspace of $\mathbb{R}^n$ orthogonal to $C(X)$

Let us study the corresponding random quantity $\underline{E}$

$$\underline{E} = (I_n - P)\underline{Y}$$
$$= (I_n - P)(X\underline{\beta} + \underline{\varepsilon})$$
$$= \underbrace{\cancel{(I_n - P)X\underline{\beta}}}_{=0} + (I_n - P)\underline{\varepsilon} = (I_n - P)\underline{\varepsilon} \qquad \underline{\varepsilon} \sim N(\underline{0}, \sigma^2 I_n)$$

$$(I_n - P)X\underline{\beta} = X\underline{\beta} - PX \cdot \underline{\beta} = \underline{0}$$

indeed, $(I_n - P)X$ is the projection of X on the space $\perp C(X)$

$$PX = X(X^T X)^{-1} X^T X = X$$

Hence, $\underline{E} = (I_n - P)\underline{\varepsilon}$ $\qquad \underline{\varepsilon} \sim N_n(\underline{0}, \sigma^2 I_n)$

linear combination of a Gaussian is Gaussian $\to$ $\underline{E} \sim N_n(\dots, \dots)$

$$\mathbb{E}[\underline{E}] = (I_n - P)\underline{0} = \underline{0}$$
$$\mathrm{var}(\underline{E}) = (I_n - P)\,\mathrm{var}(\underline{\varepsilon})(I_n - P)^T = \sigma^2 (I_n - P)(I_n - P)^T = \sigma^2 (I_n - P)$$

$$\Rightarrow \quad \underline{E} \sim N_n(\underline{0}, (I_n - P)\sigma^2)$$

(i.e. $\mathrm{var}(E_i) = \sigma^2 (I_n - P)_{(ii)}$ $\to$ not homoscedastic)

• DISTRIBUTION OF $\hat{\Sigma}^2$

Preliminary result

If $\underline{z} \sim N_d(\underline{0}, \sigma^2 I_n)$; B a $(d \times d)$ matrix, symmetric, idempotent, with rank k $(1 \leq k \leq d)$

$$\Rightarrow \quad Q = \frac{1}{\sigma^2} \underline{z}^T B \underline{z} \qquad Q \sim \chi^2_k$$

$$\hat{\Sigma}^2 = \frac{1}{n} \underline{E}^T \underline{E}$$

$$\Rightarrow \quad n\hat{\Sigma}^2 = \underline{E}^T \underline{E} = ((I_n - P)\underline{\varepsilon})^T ((I_n - P)\underline{\varepsilon})$$
$$= \underline{\varepsilon}^T (I_n - P)\underline{\varepsilon} \qquad \text{where } (I_n - P) \text{ is symmetric, idempotent, rank } (n-p)$$

$\Rightarrow$ applying the result we get

$$\frac{1}{\sigma^2} \cdot \underline{\varepsilon}^T (I_n - P)\underline{\varepsilon} \sim \chi^2_{n-p} \quad \Rightarrow \quad \frac{\underline{E}^T \underline{E}}{\sigma^2} = \frac{n\hat{\Sigma}^2}{\sigma^2} \sim \chi^2_{n-p}$$

$n - p = \mathrm{rank}(I_n - P)$
$= \#\text{units} - \#\text{covariates}$

Hence, we also find that $\mathbb{E}\left[\frac{n\hat{\Sigma}^2}{\sigma^2}\right] = n - p \Rightarrow \mathbb{E}[\hat{\Sigma}^2] = \frac{\sigma^2}{n}(n-p)$ biased

As usual, we obtain an unbiased estimator as

$$S^2 = \frac{\underline{E}^T \underline{E}}{n-p} = \frac{n\hat{\Sigma}^2}{n-p} \quad \text{with} \quad \frac{(n-p)S^2}{\sigma^2} \sim \chi^2_{n-p} \quad \Rightarrow \quad \mathbb{E}[\hat{S}^2] = \sigma^2$$

moreover, $\hat{\underline{\beta}} \perp\!\!\!\perp \hat{\Sigma}^2$ and $\hat{\underline{\beta}} \perp\!\!\!\perp S^2$