7 oct 2025 - lec. 3

GAUSSIAN SIMPLE LINEAR REGRESSION

Assume that on n statistical units we observe (y_i, x_i), $i=1,\dots,n$.
We assume that each y_i is realization of a random variable Y_i, and that $Y_1,\dots,Y_n$ are independent.
We only consider one covariate x_i, $i=1,\dots,n$.

Consider the model
$$Y_i = \beta_1 + \beta_2 x_i + \varepsilon_i \qquad i=1,\dots,n$$

HYPOTHESES:

$$\left.\begin{array}{ll} 1. & \mathbb{E}[\varepsilon_i] = 0 \quad i=1,\dots,n \\ 2. & Var(\varepsilon_i) = \sigma^2 \quad \text{for all } i=1,\dots,n \\ 3. & cov(\varepsilon_i, \varepsilon_k) = 0 \quad i \neq k;\ i,k=1,\dots,n \end{array}\right\} \text{hyp. from last time}$$

\+ 4. ε_i have Gaussian distribution
$$\Rightarrow \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2) \qquad i=1,\dots,n$$

$Y_i = \beta_1 + \beta_2 x_i + \varepsilon_i$ $\Rightarrow$ the normal distribution is closed w.r.t. linear transformations
$\Rightarrow Y_i \sim N(\beta_1 + \beta_2 x_i, \sigma^2)$ independent (but not identically distributed)

Now we have distributive assumptions, hence we can derive the estimators for $\beta_1, \beta_2, \sigma^2$ using the maximum likelihood method.

here, the parameters are $(\beta_1, \beta_2, \sigma^2)$ $\Rightarrow$ parameter space $\Theta = \mathbb{R}^2 \times (0, +\infty)$
sample space $\mathcal{Y} = \mathbb{R}^n$

likelihood function $L(\theta) \propto f(y_1,\dots,y_n;\theta) \overset{ind}{=} \prod_{i=1}^{n} f(y_i;\theta)$

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(y_i - \beta_1 - \beta_2 x_i)^2\right\}$$
$$= \cancel{(2\pi)^{-\frac{n}{2}}} (\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)^2\right\}$$

loglikelihood $\ell(\theta) = \log L(\theta)$
$$= -\frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)^2$$

score function $\ell_*(\theta) = \left[\frac{\partial \ell(\theta)}{\partial \theta_1}, \dots, \frac{\partial \ell(\theta)}{\partial \theta_q}\right]$ (here, $q=3$)

$$\begin{cases} \frac{\partial}{\partial \beta_1}\ell(\theta) = -\frac{1}{\cancel{2}\sigma^2}\sum_{i=1}^{n}(-\cancel{2})(y_i - \beta_1 - \beta_2 x_i) = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i) \\ \frac{\partial}{\partial \beta_2}\ell(\theta) = -\frac{1}{\cancel{2}\sigma^2}\sum_{i=1}^{n}(-x_i)\cancel{2}(y_i - \beta_1 - \beta_2 x_i) = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)x_i \\ \frac{\partial}{\partial \sigma^2}\ell(\theta) = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)^2 \end{cases}$$

the MLE is found as $\hat\theta$ s.t. $\ell_*(\hat\theta) = 0$

$$\begin{cases} \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i) = 0 & \rightsquigarrow \sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i) = 0 \quad (1) \\ \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)x_i = 0 & \rightsquigarrow \sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)x_i = 0 \quad (2) \\ -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)^2 = 0 \quad (3) & \end{cases}$$

(1) and (2) are exactly the same equations we already solved using OLS
they do not depend on σ^2

hence
$$\hat\beta_1 = \bar y - \hat\beta_2 \bar x \quad \text{and} \quad \hat\beta_2 = \frac{\sum_{i=1}^{n}(x_i - \bar x)(y_i - \bar y)}{\sum_{i=1}^{n}(x_i - \bar x)^2}$$
are maximum likelihood estimates.

Solving (3)
$$-\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)^2 = 0$$
$$-\frac{1}{\cancel{2}(\sigma^2)^2}\left[n\sigma^2 - \sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)^2\right] = 0 \quad \Rightarrow \quad \hat\sigma^2 = \frac{\sum_{i=1}^{n}(y_i - \hat\beta_1 - \hat\beta_2 x_i)^2}{n} \quad \text{MLE of } \sigma^2$$

The matrix of the 2nd derivatives
$$\ell_{**}(\theta) = \left\{\frac{\partial^2 \ell(\theta)}{\partial\theta_s \partial\theta_r}\right\}_{s,r=1,2,3}$$

$$\frac{\partial^2}{\partial \beta_1^2}\ell(\theta) = -\frac{n}{\sigma^2} \qquad \frac{\partial^2}{\partial\beta_1\partial\beta_2}\ell(\theta) = -\frac{n\bar x}{\sigma^2} \qquad \frac{\partial^2}{\partial\beta_1\partial\sigma^2}\ell(\theta) = -\frac{1}{(\sigma^2)^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)$$
$$\frac{\partial^2}{\partial \beta_2^2}\ell(\theta) = -\frac{\sum_{i=1}^{n}x_i^2}{\sigma^2} \qquad \frac{\partial^2}{\partial\beta_2\partial\sigma^2}\ell(\theta) = -\frac{1}{(\sigma^2)^2}\sum_{i=1}^{n}x_i(y_i - \beta_1 - \beta_2 x_i)$$
$$\frac{\partial^2}{\partial(\sigma^2)^2}\ell(\theta) = \frac{n}{2(\sigma^2)^2} - \frac{1}{(\sigma^2)^3}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)^2$$

Hence $\ell_{**}(\beta_1, \beta_2, \sigma^2)$ is the matrix

$$\begin{bmatrix} -\frac{n}{\sigma^2} & -\frac{n\bar x}{\sigma^2} & -\frac{1}{(\sigma^2)^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i) \\ -\frac{n\bar x}{\sigma^2} & -\frac{\sum_{i=1}^{n}x_i^2}{\sigma^2} & -\frac{1}{(\sigma^2)^2}\sum_{i=1}^{n}x_i(y_i - \beta_1 - \beta_2 x_i) \\ -\frac{1}{(\sigma^2)^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i) & -\frac{1}{(\sigma^2)^2}\sum_{i=1}^{n}x_i(y_i - \beta_1 - \beta_2 x_i) & \frac{n}{2(\sigma^2)^2} - \frac{1}{(\sigma^2)^3}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)^2 \end{bmatrix}$$

these are the arguments of the lik. equations (1) and (2). Hence they are $=0$ if evaluated at $(\hat\beta_1, \hat\beta_2, \hat\sigma^2)$

We need to evaluate these derivatives at $(\hat\beta_1, \hat\beta_2, \hat\sigma^2)$
Both $\frac{\partial^2}{\partial\beta_1\partial\sigma^2}\ell(\theta)$ and $\frac{\partial^2}{\partial\beta_2\partial\sigma^2}\ell(\theta)$ are $=0$ at $(\hat\beta_1, \hat\beta_2, \hat\sigma^2)$.

$$\frac{\partial^2}{\partial(\sigma^2)^2}\ell(\theta)\Big|_{\theta=\hat\theta} = \frac{n}{2(\hat\sigma^2)^2} - \frac{1}{(\hat\sigma^2)^3}\underbrace{\sum_{i=1}^{n}(y_i - \hat\beta_1 - \hat\beta_2 x_i)^2}_{=n\hat\sigma^2}$$
$$= \frac{n}{2(\hat\sigma^2)^2} - \frac{n}{(\hat\sigma^2)^2}$$
$$= -\frac{n}{2(\hat\sigma^2)^2}$$

the observed information $j(\hat\theta) = -\ell_{**}(\hat\theta)$ then is

$$j(\hat\theta) = \left[\begin{array}{cc|c} \frac{n}{\hat\sigma^2} & \frac{n\bar x}{\hat\sigma^2} & 0 \\ \frac{n\bar x}{\hat\sigma^2} & \frac{\sum_{i=1}^{n}x_i^2}{\hat\sigma^2} & 0 \\ \hline 0 & 0 & \frac{n}{2(\hat\sigma^2)^2} \end{array}\right] = \left[\begin{array}{c|c} \underset{2\times2}{A} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 \quad 0 & b \end{array}\right]$$

and it is possible to show that $(\hat\beta_1, \hat\beta_2, \hat\sigma^2)$ is a maximum.

The maximum likelihood ESTIMATORS of $(\beta_1, \beta_2, \sigma^2)$ are
$$\hat B_1 = \bar Y - \hat B_2 \bar x \quad \text{and}$$
$$\hat B_2 = \frac{\sum_{i=1}^{n}(x_i - \bar x)(Y_i - \bar Y)}{\sum_{i=1}^{n}(x_i - \bar x)^2}$$
$$\hat\Sigma^2 = \frac{\sum_{i=1}^{n}(Y_i - \hat B_1 - \hat B_2 x_i)^2}{n}$$