INFERENCE in the MULTIPLE GAUSSIAN LINEAR MODEL

we will work under the assumption that the model always includes the intercept $\underline{x}_1 = \underline{1}_n$ with β_1 the associated coefficient.

1. TEST about an individual coefficient β_j ($j = 2, \dots, p$)

assume that we want to test a single coefficient:

$$\begin{cases} H_0: \beta_j = b_j \\ H_1: \beta_j \neq b_j \end{cases}$$

In particular, we are often interested in testing the statistical significance of an individual coefficient

$$\begin{cases} H_0: \beta_j = 0 \\ H_1: \beta_j \neq 0 \end{cases}$$

Recall:
- $\underline{\hat{\beta}} \sim N_p(\underline{\beta}, (X^TX)^{-1}\sigma^2)$
- the j-th element $\hat{\beta}_j \sim N(\beta_j, \underbrace{\sigma^2 [(X^TX)^{-1}]_{j,j}}_{var(\hat{\beta}_j)})$
- $\frac{n\hat{\Sigma}^2}{\sigma^2} \sim \chi^2_{n-p}$
- $\frac{(n-p)S^2}{\sigma^2} \sim \chi^2_{n-p}$
- $\underline{\hat{\beta}} \perp\!\!\!\perp \hat{\Sigma}^2$ and $\underline{\hat{\beta}} \perp\!\!\!\perp S^2$

1) We need to define a TEST STATISTIC with known distribution under H_0.

$$\frac{\hat{\beta}_j - b_j}{\sqrt{var(\hat{\beta}_j)}} = \frac{\hat{\beta}_j - b_j}{\sqrt{\sigma^2 [(X^TX)^{-1}]_{j,j}}} \overset{H_0}{\sim} N(0,1)$$ but it depends on the unknown σ^2 (hence we can't use it)

we consider instead

$$T_j = \frac{\hat{\beta}_j - b_j}{\sqrt{S^2 [(X^TX)^{-1}]_{j,j}}} = \frac{\hat{\beta}_j - b_j}{\sqrt{\hat{var}(\hat{\beta}_j)}} =$$

$$= \frac{\hat{\beta}_j - b_j}{\sqrt{\frac{S^2}{\sigma^2} var(\hat{\beta}_j)}} = \frac{\overbrace{\frac{\hat{\beta}_j - b_j}{\sqrt{v(\hat{\beta}_j)}}}^{\sim N(0,1)}}{\sqrt{\frac{S^2}{\sigma^2}} \sim \sqrt{\frac{\chi^2_{n-p}}{(n-p)}}}$$

$$\hat{var}(\hat{\beta}_j) = S^2[(X^TX)^{-1}]_{jj} \cdot \frac{\sigma^2}{\sigma^2}$$

$$= (\sigma^2[(X^TX)^{-1}]_{jj}) \cdot \frac{S^2}{\sigma^2} = var(\hat{\beta}_j) \cdot \frac{S^2}{\sigma^2}$$ general expression

$$\Rightarrow \quad T_j = \frac{\hat{\beta}_j - b_j}{\sqrt{\hat{var}(\hat{\beta}_j)}} \overset{H_0}{\sim} t_{n-p}$$

in the simple lm we had $(t-2)$ degrees of freedom. Indeed $p=2$ for the simple lm $X = [\underline{1}\ \underline{x}]$

2) With the data, I compute the OBSERVED VALUE OF THE TEST t_j^{obs}

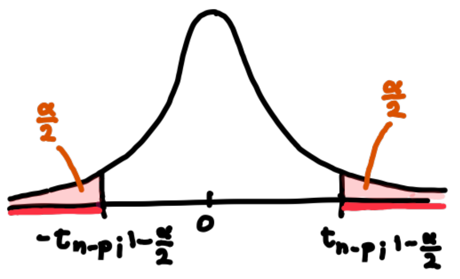
3) We study the partition of the sample space into the REJECT and ACCEPTANCE REGION:
As for the simple linear model, large values of the test (in absolute value) lead to rejecting the null hypothesis (if H_0 is not true, $\hat{\beta}_j$ will be very different from b_j, hence $|\hat{\beta}_j - b_j| \gg 0$ and also $|t^{obs}| \gg 0$).

Hence: $\mathcal{A} = (-k, k)$
$\mathcal{R} = (-\infty; -k) \cup (k, +\infty)$

4) We conclude the test

4a) FIXED SIGNIFICANCE LEVEL α

$$\mathbb{P}_{H_0}(T_j \in \mathcal{R}) = \mathbb{P}_{H_0}(|T_j| > t_{n-p;1-\frac{\alpha}{2}}) = \alpha$$



i.e.

$$\mathcal{R} = (-\infty, -t_{n-p;1-\frac{\alpha}{2}}) \cup (t_{n-p;1-\frac{\alpha}{2}}, +\infty)$$

and reject H_0 if $t^{obs} \in \mathcal{R}$

4b) p-value $= \mathbb{P}_{H_0}(|T_j| > |t_j^{obs}|) =$
$= 2\,\mathbb{P}_{H_0}(T_j > |t_j^{obs}|)$ with $T_j \sim t_{n-p}$