Exercise 2: mother and daughter data

(x_i, y_i) $\quad i = 1, \dots, n \qquad n = 11$

a) the model is $Y_i = \beta_1 + \beta_2 x_i + \varepsilon_i \qquad i = 1, \dots, n$ with $\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$, $\sigma^2 > 0$.
moreover, we need to have $S_x^2 \neq 0$, which is satisfied here.

b) We know that the mle are

$\hat{\beta}_1 = \bar{y} - \hat{\beta}_2 \bar{x}$

$\hat{\beta}_2 = \frac{S_{xy}}{S_x^2}$

$$\Rightarrow \hat{\beta}_2 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{\sum_{i=1}^n x_i y_i - \overbrace{\bar{x} \sum_{i=1}^n y_i}^{n\bar{x}\bar{y}} - \overbrace{\bar{y} \sum_{i=1}^n x_i}^{n\bar{x}\bar{y}} + n\bar{x}\bar{y}}{\sum_{i=1}^n x_i^2 + n\bar{x}^2 - \underbrace{2\bar{x} \sum_{i=1}^n x_i}_{= 2n\bar{x}^2}} = \frac{\sum_{i=1}^n x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^n x_i^2 - n\bar{x}^2}$$

$$= \frac{284335.1 - 11 \cdot 159.76 \cdot 161.49}{281940.6 - 11 \cdot 159.76^2} = \frac{539.033}{1184.766} = 0.454$$

$$\Rightarrow \hat{\beta}_1 = 161.49 - 0.454 \cdot 159.76 = 88.95$$

c) compute $s^2 = \frac{1}{(n-2)} \sum_{i=1}^n (y_i - \hat{y}_i) = \frac{1}{(n-2)} \sum_{i=1}^n e_i^2$

we need the residuals $\Rightarrow$ we need the predicted values

$\hat{y}_i = \hat{\beta}_1 + \hat{\beta}_2 x_i = 88.95 + 0.454 \cdot x_i$

we have to compute it for all $i = 1, \dots, 11$

$i=1)$ $\hat{y}_1 = 88.95 + 0.454 \cdot 153.7 = 158.73$

$i=2)$ $\hat{y}_2 = 88.95 + 0.454 \cdot 156.7 = 160.09$

$i=3)$ ...

...

we do this count for all i.

Then, the residuals are

$i=1)$ $e_1 = y_1 - \hat{y}_1 = 163.1 - 158.73 = 4.35$

$i=2)$ $e_2 = y_2 - \hat{y}_2 = 159.5 - 160.09 = -0.60$

$i=3)$ ...

...

We obtain s^2 as $\quad s^2 = \frac{1}{9} \left(4.35^2 + (-0.60)^2 + \dots + \dots \right) = 7.36$

d) $\begin{cases} H_0: \beta_2 = 1 \\ H_1: \beta_2 \neq 1 \end{cases}$

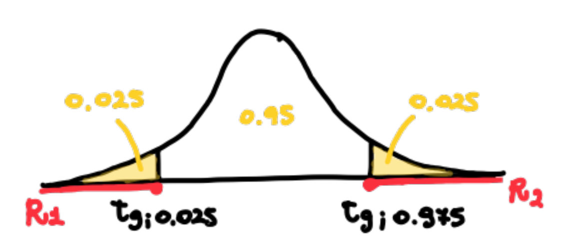
under H_0 ("if H_0 true"), the value assumed for β_2 is 1.

Hence, under H_0, $\hat{\beta}_2 \overset{H_0}{\sim} N(1, \underbrace{var(\hat{\beta}_2)}_{\text{contains } \sigma^2 \text{, unknown}})$

test statistic: $\quad T = \frac{\hat{\beta}_2 - 1}{\sqrt{\hat{var}(\hat{\beta}_2)}} \overset{H_0}{\sim} t_{n-2} = t_9$

We have a two-side test $\Rightarrow$ two-side reject region

$R_0 = (-\infty; a) \cup (b; +\infty) = R_1 \cup R_2$



If we consider a significance level $\alpha = 0.05$,

$a = -t_{9;0.025} \qquad b = t_{9;0.975}$

Moreover, t distribution is symmetric $\Rightarrow a = -t_{9;0.975}$

$R_0 = (-\infty; -2.26) \cup (2.26; +\infty)$

With the data:

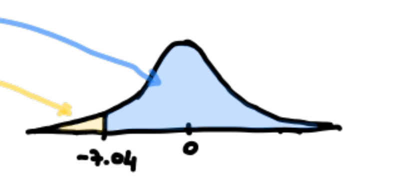
$t^{obs} = \frac{\hat{\beta}_2 - 1}{\sqrt{\hat{var}(\hat{\beta}_2)}}$ $\qquad$ we need the estimate $\hat{var}(\hat{\beta}_2) = \frac{s^2}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{7.36}{1184.766} = 0.0063$

$= \frac{0.454 - 1}{\sqrt{0.006}} = -7.04$

$t^{obs} \in R_0 \Rightarrow$ we reject H_0 at a level $\alpha = 0.05$

If we want the p-value

$\alpha^{obs} = 2 \min \{ \mathbb{P}_{H_0}(T \geq t^{obs}); \mathbb{P}_{H_0}(T \leq t^{obs}) \}$



$= 2\, \mathbb{P}_{H_0}(T \leq t^{obs})$

$= 2\, \mathbb{P}_{H_0}(T \leq -7.04) \qquad$ where $T \overset{H_0}{\sim} t_9$

$= 2\, \mathbb{P}_{H_0}(T \geq +7.04)$

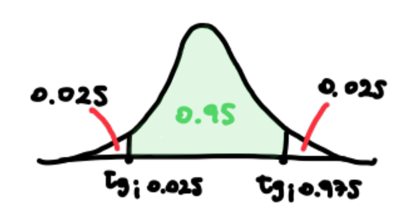
$= 2 \underbrace{\left(1 - \underbrace{\mathbb{P}_{H_0}(T \leq 7.04)}_{> 0.999}\right)}_{< 0.001} < 0.002$

e) confidence interval with $1 - \alpha = 0.95$

we start with β_1

we know that $\frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{var}(\hat{\beta}_1)}} \sim t_9$ $\qquad$ true value, unknown

Hence $\quad \mathbb{P}\left(t_{9;0.025} \leq \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{var}(\hat{\beta}_1)}} \leq t_{9;0.975} \right)$



$\mathbb{P}\left(t_{9;0.025} \cdot \sqrt{\hat{var}(\hat{\beta}_1)} \leq \hat{\beta}_1 - \beta_1 \leq t_{9;0.975} \sqrt{\hat{var}(\hat{\beta}_1)} \right) = 0.95$

$\mathbb{P}\left(-\hat{\beta}_1 + t_{9;0.025} \cdot \sqrt{\hat{var}(\hat{\beta}_1)} \leq -\beta_1 \leq -\hat{\beta}_1 + t_{9;0.975} \sqrt{\hat{var}(\hat{\beta}_1)} \right) = 0.95$

$\mathbb{P}\left(\hat{\beta}_1 - t_{9;0.975} \sqrt{\hat{var}(\hat{\beta}_1)} \leq \beta_1 \leq \hat{\beta}_1 - \underbrace{t_{9;0.025}}_{= -t_{9;0.975}} \cdot \sqrt{\hat{var}(\hat{\beta}_1)} \right) = 0.95$

$\mathbb{P}\left(\hat{\beta}_1 - t_{9;0.975} \sqrt{\hat{var}(\hat{\beta}_1)} \leq \beta_1 \leq \hat{\beta}_1 + t_{9;0.975} \cdot \sqrt{\hat{var}(\hat{\beta}_1)} \right) = 0.95$

With the data, $\hat{\beta}_1 = 88.95$

$\hat{var}(\hat{\beta}_1) = s^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right) = 7.36 \left(\frac{1}{11} + \frac{159.76^2}{1184.766} \right) = 159.22$

$\sqrt{\hat{var}(\hat{\beta}_1)} = 12.61$

Hence $\quad \beta_1 \in \left(\hat{\beta}_1 \pm t_{9;0.975} \cdot \sqrt{\hat{var}(\hat{\beta}_1)} \right)$

$\in (88.95 \pm 2.26 \cdot 12.61)$

$\in (60.45; 117.44)$

Similarly, for β_2, ...

f) $SST = \sum_{i=1}^n (y_i - \bar{y})^2 = \sum_{i=1}^n y_i^2 - n\bar{y}^2 = 287179.3 - 11 \cdot 161.49^2 = 306.809$

$SSE = \sum_{i=1}^n (y_i - \hat{y}_i)^2 = (n-2)\, s^2 = 9 \cdot 7.36 = 66.24$

$SSR = \sum_{i=1}^n (\hat{y}_i - \bar{y})^2 = SST - SSE = 240.56$

$R^2 = \frac{SSR}{SST} = \frac{240.56}{306.809} = 0.784$

$\rho_{XY} = \frac{S_{XY}}{S_X S_Y} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2 \sum_{i=1}^n (y_i - \bar{y})^2}} = \frac{539.033}{\sqrt{1184.766 \cdot 306.809}} = 0.894$

and $\rho_{XY}^2 = 0.79 = R^2$

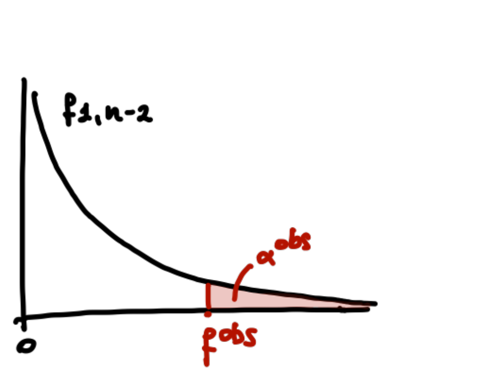
g) $\begin{cases} H_0: R^2 = 0 \\ H_1: R^2 > 0 \end{cases}$

We use the test statistic

$F = \frac{R^2}{1 - R^2} (n-2) \overset{H_0}{\sim} F_{1,n-2} = F_{1,9}$

With the data we obtain the observed value of the test,

$f^{obs} = \frac{0.784}{1 - 0.784} \cdot 9 = 32.81$



The p-value is the probability of observing "more extreme" values than f^{obs}, "more against H_0".

The p-value is $\alpha^{obs} = \mathbb{P}_{H_0}(F \geq f^{obs})$

$= \mathbb{P}_{H_0}(F \geq 32.81) < 0.001$

We reject H_0 for $\alpha = 0.05, 0.01, 0.001$.

The model is useful to explain the variability of y

We know that the tests

$\begin{cases} H_0: \beta_2 = 0 \\ H_1: \beta_2 \neq 0 \end{cases}$ and $\begin{cases} H_0: R^2 = 0 \\ H_1: R^2 \neq 0 \end{cases}$

are equivalent.

h) In the first case, we use the test statistic

$$T = \frac{\hat{\beta}_2}{\sqrt{\frac{S^2}{\sum_{i=1}^n (x_i - \bar{x})^2}}} \overset{H_0}{\sim} t_{n-2}$$

in the second, we use

$$F = \frac{R^2}{1 - R^2} (n-2) = \left(\frac{SST}{SSE} - 1 \right) (n-2) \overset{H_0}{\sim} F_{1,n-2}$$

the equivalence holds because

$F = T^2$

With the data

$f^{obs} = 32.81$

$t^{obs} = \frac{0.45}{\sqrt{0.0063}} = 5.66 \Rightarrow (t^{obs})^2 = 32.14$