EXERCISE 1 EXAM 25/01/2024

$(y_1, \dots, y_5)$ sample from $Y_i \sim Pois(e^{\beta_1})$ indep. $i = 1, \dots, 5$

$(y_6, \dots, y_{10})$ sample from $Y_i \sim Pois(e^{\beta_1 + \beta_2})$ indep. $i = 6, \dots, 10$

a) • $Y_i \sim Poisson(\mu_i)$ $i = 1, \dots, 10$ indep.

• $\eta_i = \beta_1 + \beta_2 x_i$ with $x_i = \begin{cases} 0 & i = 1, \dots, 5 \\ 1 & i = 6, \dots, 10 \end{cases}$

• $\log \mu_i = \eta_i$

b) $p(y_i ; \mu_i) = e^{-\mu_i} \mu_i^{y_i} \cdot \frac{1}{y_i!}$

$$p(y_1, \dots, y_{10} ; \underline{\mu}) = \prod_{i=1}^{10} \left(e^{-\mu_i} \mu_i^{y_i} \frac{1}{y_i!} \right) \qquad \text{with } \mu_i = e^{\tilde{x}_i^T \beta}$$

$$L(\underline{\beta}) \propto \prod_{i=1}^{10} e^{-\mu_i} \mu_i^{y_i} = \prod_{i=1}^{n} e^{-e^{\tilde{x}_i^T \beta}} e^{\tilde{x}_i^T \beta \, y_i}$$

$$\ell(\underline{\beta}) = \sum_{i=1}^{n} - e^{\tilde{x}_i^T \beta} + y_i \tilde{x}_i^T \beta$$

here, $\tilde{x}_i^T \beta = \beta_1 + \beta_2 x_i$ with x_i dummy variable, hence:

$$\begin{aligned} \ell(\beta_1, \beta_2) &= \sum_{i=1}^{10} \left\{ - e^{\beta_1 + \beta_2 x_i} + y_i (\beta_1 + \beta_2 x_i) \right\} \\ &= - \sum_{i=1}^{10} e^{\beta_1 + \beta_2 x_i} + \beta_1 \sum_{i=1}^{10} y_i + \beta_2 \sum_{i=1}^{10} x_i y_i \\ &= - \sum_{i=1}^{5} e^{\beta_1} - \sum_{i=6}^{10} e^{\beta_1 + \beta_2} + \beta_1 \sum_{i=1}^{10} y_i + \beta_2 \sum_{i=6}^{10} y_i \\ &= -5e^{\beta_1} - 5e^{\beta_1 + \beta_2} + \beta_1 \sum_{i=1}^{10} y_i + \beta_2 \sum_{i=6}^{10} y_i \end{aligned}$$

$$\ell_*(\beta_1, \beta_2) = \begin{cases} \dfrac{\partial \ell(\beta)}{\partial \beta_1} = -5e^{\beta_1} - 5e^{\beta_1 + \beta_2} + \sum_{i=1}^{10} y_i \\ \dfrac{\partial \ell(\beta)}{\partial \beta_2} = -5e^{\beta_1 + \beta_2} + \sum_{i=6}^{10} y_i \end{cases}$$

score function

$$\ell_{**}(\beta_1, \beta_2) = \begin{bmatrix} -5e^{\beta_1} - 5e^{\beta_1 + \beta_2} & -5e^{\beta_1 + \beta_2} \\ -5e^{\beta_1 + \beta_2} & -5e^{\beta_1 + \beta_2} \end{bmatrix} = -5 \begin{bmatrix} e^{\beta_1} + e^{\beta_1 + \beta_2} & e^{\beta_1 + \beta_2} \\ e^{\beta_1 + \beta_2} & e^{\beta_1 + \beta_2} \end{bmatrix}$$

observed info is $j(\underline{\beta}) = -\ell_{**}(\underline{\beta})$

The MLE can be found in closed-form by noticing that:

for sample 1 $\underline{y}_1 = (y_1, \dots, y_5)$ the expected value is $\mathbb{E}[Y_i] = \mu_1 = e^{\beta_1}$

for sample 2 $\underline{y}_2 = (y_6, \dots, y_{10})$ the expected value is $\mathbb{E}[Y_i] = \mu_2 = e^{\beta_1 + \beta_2}$

The function from (β_1, β_2) to (μ_1, μ_2) is bijective ($\Rightarrow$ invertible)

$$\begin{cases} \mu_1 = e^{\beta_1} \\ \mu_2 = e^{\beta_1 + \beta_2} \end{cases} \Leftrightarrow \begin{cases} \beta_1 = \log \mu_1 \\ \beta_1 + \beta_2 = \log \mu_2 \end{cases} \Rightarrow \beta_2 = \log \mu_2 - \log \mu_1$$

I can obtain the MLE $(\hat{\mu}_1, \hat{\mu}_2)$ and obtain $(\hat{\beta}_1, \hat{\beta}_2) = f(\hat{\mu}_1, \hat{\mu}_2)$

The MLE of μ_1 and μ_2 are:

$$\hat{\mu}_1 = \bar{y}_1 = \frac{1}{5} \sum_{i=1}^{5} y_i \qquad \text{sample mean of } \underline{y}_1$$

$$\hat{\mu}_2 = \bar{y}_2 = \frac{1}{5} \sum_{i=6}^{10} y_i \qquad \text{sample mean of } \underline{y}_2$$

$$\Rightarrow \begin{cases} \hat{\beta}_1 = \log \hat{\mu}_1 \\ \hat{\beta}_2 = \log \hat{\mu}_2 - \log \hat{\mu}_1 \end{cases} \Rightarrow \begin{cases} \hat{\beta}_1 = \log \bar{y}_1 \\ \hat{\beta}_2 = \log \bar{y}_2 - \log \bar{y}_1 \end{cases}$$

Alternatively, one can solve the likelihood equations

$$\begin{cases} -5e^{\beta_1} - 5e^{\beta_1 + \beta_2} + \sum_{i=1}^{10} y_i = 0 & \Rightarrow \quad 5e^{\beta_1 + \beta_2} = -5e^{\beta_1} + \sum_{i=1}^{10} y_i \\ -5e^{\beta_1 + \beta_2} + \sum_{i=6}^{10} y_i = 0 & \Rightarrow \quad 5e^{\beta_1 + \beta_2} = \sum_{i=6}^{10} y_i \end{cases}$$

$$\Rightarrow -5e^{\beta_1} + \sum_{i=1}^{10} y_i = \sum_{i=6}^{10} y_i$$

$$e^{\beta_1} = \frac{1}{5} \left(\sum_{i=1}^{10} y_i - \sum_{i=6}^{10} y_i \right) = \frac{1}{5} \sum_{i=1}^{5} y_i = \bar{y}_1 \quad \Rightarrow \quad \hat{\beta}_1 = \log \bar{y}_1$$

From eq. 2: $5e^{\beta_1} e^{\beta_2} = \sum_{i=6}^{10} y_i \Rightarrow e^{\beta_2} = \frac{1}{5} \sum_{i=6}^{10} y_i \, e^{-\beta_1}$

$$\beta_2 = \log \bar{y}_2 - \beta_1 \quad \Rightarrow \quad \hat{\beta}_2 = \log \bar{y}_2 - \log \bar{y}_1$$

c) $$\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} \dot{\sim} N_2 \left(\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, j(\hat{\underline{\beta}})^{-1} \right)$$

the marginal $\hat{\beta}_1 \dot{\sim} N(\beta_1, [j(\hat{\underline{\beta}})^{-1}]_{(1,1)})$

↳ element in position (1, 1)

d) we already noticed that

for sample 1 $\mathbb{E}[Y_i] = \mu_1 = e^{\beta_1}$

for sample 2 $\mathbb{E}[Y_i] = \mu_2 = e^{\beta_1 + \beta_2}$

$$\frac{\mu_2}{\mu_1} = \frac{e^{\cancel{\beta_1} + \beta_2}}{\cancel{e^{\beta_1}}} = e^{\beta_2} \quad \Rightarrow \quad \beta_2 = \log \frac{\mu_2}{\mu_1} = \log \frac{\mathbb{E}[Y_i \mid \text{sample } 2]}{\mathbb{E}[Y_i \mid \text{sample } 1]}$$

Hence β_2 represents the log of the ratio between the expected counts of the two samples.

Or, noting that $\mu_2 = e^{\beta_2} \cdot \mu_1$:

the expected counts of sample 2 are obtained by multiplying the expected counts of sample 1 by a coefficient e^{β_2}.

e) The saturated model is a model with n parameters $\mu_1, \dots, \mu_n$.

In this case, I have one parameter for each observation and the estimates are $\hat{\mu}_i^S = y_i \ \forall i$

The loglikelihood for an individual observation is

$$\ell(\mu_i^S) = -\mu_i^S + y_i \log \mu_i^S$$

Hence the max of the loglikelihood is

$$\tilde{\ell}(\text{saturated}) = \sum_{i=1}^{10} \left\{ -\hat{\mu}_i^S + y_i \log \hat{\mu}_i^S \right\} = \sum_{i=1}^{10} \left\{ -y_i + y_i \log y_i \right\}$$